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An Interesting Representation of Lie Algebras of Linear Groups

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Abstract

I have presented a means of getting a representation space of a general linear group of n dimensions in terms of homogeneous functions of n, n-dimensional vectors. Except in particular cases, the representation is of the Lie algebra, rather than the group. A general formalism is set up to evaluate the Casimir operators of the Lie algebra of the group in terms of the degrees of homogeneity of the functions (which are eigenfunctions of the Casimir operators) in the n variables. It is noticed that the Casimir operators exhibit certain symmetries in these degrees of homogeneity which relate different representations having the same eigenvalues for the Casimir operators. Contour integral formulas that enable one to pass from one such representation to another are presented. An expression for the eigenvalues of a general Casimir operator in terms of the degree of homogeneity is presented.

1. *Introduction*

The work for this paper arises from some work done on twistors (Penrose, 1967), based on certain suggestions made by Prof. R. Penrose. Twistors are "vectors" of the representation space of $U(2, 2)$. Now $SU(2, 2)$ is locally isomorphic to the 15-parameter conformal group of compactified Minkowski space and to $O(2, 4)$. The results for twistors are analogous to the results for two-component spinors – "vectors" of $U(2)$, $SU(2)$ being locally isomorphic to $O(3)$. I have attempted to set up a formalism in which the Casimir operators of the Lie algebra of any linear group can be easily expressed and in fact may be read off from a general formula. The results apply equally to unitary and other groups, as well as to linear groups. The reason why the Lie algebra is used instead of the group will become clear later. The representations will be given in terms of multivariable functions that are homogeneous in all the variables. The variables are "vectors" of the representation space of the algebra. It will be easily seen that the number of variables required is the dimension of the representation space.

The "canonical generators" are defined in terms of the variables and derivatives with respect to those variables. Contractions of the canonical generators, among themselves, so that no free indices are left, give the Casimir operators. The number of linearly independent Casimir operator is the rank of the group. The functions used for getting representations of the algebra are eigenfunctions of the Casimir operators. The original interest in these functions was because they tied in with certain expressions for the solutions of the zero rest mass field equations (Penrose, 1969). The Casimir operators are found to possess certain symmetries using one which can give a general expression for the eigenvalues of any Casimir operator of the Lie algebra of a linear group.

The methods used here are inspired by the method of Gel'fand et al. (1966) to get representations of $SL(2, C)$ and of $GL(2, C)$.

The paper is presented in the following way. First, representations of $GL(n, C)$ are dealt with. Then a theorem is proved for the functions giving representations of the Lie algebra of $GL(n, C)$. Then the Casimir operators are defined and worked out for $GL(2, C)$, $GL(3, C)$ and $GL(4, C)$. It is observed that these Casimir operators possess certain symmetries. It is proved that all Casimir operators of the Lie algebra (*not* of the group) possess this symmetry.

2. *Representations of GL(n, C)*

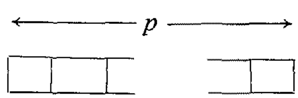
Consider a function $f(\xi^a)$, homogeneous of degree p in a complex n-component vector ξ^a, i.e.,

$$f(A\xi^a) = A^p f(\xi^a) \tag{2.1}$$

In particular, $f(\xi^a)$ might be expressible as a polynomial of degree p in ξ^a, i.e.,

$$f(\xi^a) = f_{a\cdots c}\xi^a \cdots \xi^c \tag{2.2}$$

where $f_{a\cdots c} = f_{(a\cdots c)}$. Such $f_{a\cdots c}$ can be represented by the Young tableau



All such $f_{a\cdots c}$'s form a $(p + 1)$-dimensional, irreducible representation space for $GL(n, C)$. Such polynomial expressions are nonsingular for all ξ^a. In general, however, $f(\xi^a)$ is nonsingular only over some domain, being singular somewhere outside that domain. In such a case $f(\xi^a)$ will give a representation of the Lie algebra of the group, rather than the group itself, because the domain is shifted by the action of the group but is not shifted by the action of the algebra. An $f(\xi^a)$ can be constructed such that p is a positive integer, but $f(\xi^a)$ cannot be expressed as a polynomial in ξ^a, e.g., with $p = +1$,

$$f(\xi^a) = L_{ab}\xi^a\xi^b/K_c\xi^c \tag{2.3}$$

cannot be expressed as a polynomial in general. Such functions span an infinite-dimensional representation space of the Lie algebra of $GL(n, C)$. If p is not a positive integer, $f(\xi^a)$ leads to an irreducible representation space.

To see that $f(\xi^a)$ gives a representation of the Lie algebra of the group, first consider two elements of the group $\mathscr{U}$ and $\mathscr{T}$. If we want to expand close to the identity, we can put $\mathscr{U} = \mathbf{1} + \epsilon\mathbf{U}$ and $\mathscr{T} = \mathbf{1} + \epsilon\mathbf{T}$, where ϵ is infinitesimal. Let $\mathbf{U}$ and $\mathbf{T}$ be given by $\mathbf{U} : \boldsymbol{\xi} = U^a_b \xi^b$ and $\mathbf{T} : \boldsymbol{\xi} = T^a_b \xi^b$. Then

$$(\underset{\mathscr{U}}{R}f)(\boldsymbol{\xi}) = f(\xi^a) + \epsilon U^a_b \xi^b f/\partial\xi^a, \qquad \text{i.e., } \underset{\mathbf{U}}{R}F(\boldsymbol{\xi}) = U^a_b \xi^b f/\partial\xi^a$$

$$(\underset{\mathscr{T}}{R}f)(\boldsymbol{\xi}) = f(\xi^a) + \epsilon T^a_b \xi^b \partial f/\partial\xi^a, \qquad \text{i.e., } \underset{\mathbf{T}}{R}F(\boldsymbol{\xi}) = T^a_b \xi^b \partial f/\partial\xi^a$$

$$\therefore (\underset{\mathbf{U}}{R} + \underset{\mathbf{T}}{R})f(\boldsymbol{\xi}) = \xi^b(U^a_b + T^a_b)(\partial f/\partial\xi^a)(\xi^c) = \boldsymbol{\xi}\cdot(\mathbf{U}+\mathbf{T})\cdot(\partial f/\partial\boldsymbol{\xi})(\boldsymbol{\xi})$$

$$\therefore \quad [\underset{\mathbf{U}}{R}, \underset{\mathbf{T}}{R}]f(\boldsymbol{\xi}) = \xi^a(U^b_a T^c_b - T^b_a U^c_b)(\partial f/\partial\xi^c)(\xi^d)$$

$$= \boldsymbol{\xi}\cdot(\mathbf{U}\cdot\mathbf{T} - \mathbf{T}\cdot\mathbf{U})\cdot(\partial f/\partial\boldsymbol{\xi})(\boldsymbol{\xi})$$

It can easily be checked that these functions satisfy all the requirements of a representation space, the operators being represented by $\boldsymbol{\xi}\cdot\mathbf{U}\cdot(\partial/\partial\boldsymbol{\xi})$. The representation is of an algebra, instead of being of a group, because the function will often be singular for some value of ξ^a (for reasons which become apparent later). Thus the action of the group shifts the domain over which $f(\xi^a)$ is nonsingular. Now if the transformations of ξ^a are infinitesimal and the function is nonsingular over some domain, the transformed function will be nonsingular over the same domain. Thus representations of the algebra are valid, but of the group they are not valid.

I shall now prove a theorem for these functions, taking them to be of n variables for $GL(n, C)$. I shall explain what they represent later.

Theorem 1: For functions, homogeneous in n (n-dimensional) variables $W^a, X^a, \ldots, Z^a$, the following statements are equivalent:

1. $f(W^a, X^b, \ldots, Z^d) = f(W^a + \lambda X^a, X^b, \ldots, Z^d)$

$$= f(W^a + \lambda Z^a, X^b, \ldots, Z^d) \tag{2.4a}$$

2. $X^a\, \partial f/\partial W^a = \cdots = Z^a\, \partial f/\partial W^a = 0$ (2.4b)
3. $f(W^a, X^b, \ldots, Z^d) = g(W^{[a}X^b \ldots Z^{d]}, X^k, \ldots, Z^m)$ (2.4c)

Proof: (i) We know that

$$(\partial/\partial\lambda)f(W^a + \lambda X^a, X^b, \ldots, Z^d) = X^e(\partial/\partial W^e)f(W^a + \lambda X^a, X^b, \ldots, Z^d)$$

Now (1) implies that $\partial f/\partial\lambda = 0$. Thus we have $X^e(\partial f/\partial W^e) = 0$ and similarly up to $Z^e(\partial f/\partial W^e) = 0$. Thus (1) $\Rightarrow$ (2).

(ii) Defining $\partial f/\partial W^a = f_a$, we can write $f_a = \epsilon_{ab\cdots d}K^{b\cdots d}$ where $K^{b\cdots d} = K^{(b\cdots d)}$. Now from (2) we have

$$\epsilon_{ab\cdots d}K^{b\cdots d}X^a = \cdots = \epsilon_{ab\cdots d}K^{b\cdots d}Z^a = 0$$

Thus we have

$$X^{[a}K^{b\ldots d]} = \cdots = Z^{[a}K^{b\ldots d]} = 0$$

Thus $K^{b\cdots d}$ represents a hyperplane containing the vectors $X^a, \ldots, Z^a$. Thus $K^{b\cdots d} \propto X^{[b\cdots}Z^{d]}$. Now (3) requires that f should be constant when $W^{[a}X^b \ldots Z^{d]}$, $W^k, \ldots, Z^m$ are constant, i.e., treating $W^a, \ldots, Z^d$ as the basic variables and $\mathbf{V}$ any vector defined on the space of which $W^a, \ldots, Z^d$ are the coordinates

$$\mathbf{V}(W^{[a}X^b \cdots Z^{d]}, X^k, \ldots, Z^m) = 0 \Rightarrow \mathbf{V}f = 0$$

where we write $\mathbf{V} = V_1^a(\partial/\partial W^a) + V_2^a(\partial/\partial X^a) + \cdots + V_n^a(\partial/\partial Z^a)$. This gives us

$$(V_1^e, \ldots, V_n^e)\begin{pmatrix} \delta_e^{[a}X^b \cdots Z^{d]} & 0 & 0 & \cdots & 0 \\ W^{[a}\delta_e^b \cdots Z^{d]} & \delta_e^k & 0 & \cdots & 0 \\ W^{[a}X^b\delta_e^c \cdots Z^{d]} & 0 & \delta_e^l & \cdots & 0 \\ W^{[a}X^b \cdots Y^c\delta_e^{d]} & 0 & 0 & \cdots & \delta_e^m \end{pmatrix} = 0 \Rightarrow (V_1^e, \ldots, V_n^e)\begin{pmatrix} \partial f/\partial W^e \\ \partial f/\partial X^e \\ \vdots \\ \partial f/\partial Z^e \end{pmatrix} = 0$$

for all $\mathbf{V}$, i.e., that the column vector is linearly dependent on the square matrix. To see that (2) implies this, we reduce the matrix by subtracting the appropriate multiple of each later column from the first column, so as to obtain a diagonal matrix with δ's on the diagonal, except for the first term, which is $\delta_e^{[a}X^b \cdots Z^{d]}$ thus we require that whenever (2) is true $\partial f/\partial W^e$ is linearly dependent on $\delta_e^{[a}X^b \cdots Z^{d]}$. This has already been proved. Thus we can write $f(W^a, X^b, \ldots, Z^d) = g(W^{[a}X^b, \ldots, Z^{d]}, X^k, \ldots, Z^m)$. Thus (2) $\Rightarrow$ (3).

(iii) Clearly (3) $\Rightarrow$ (1), as any arbitrary multiple of $X^a, \ldots, Z^a$, when added to W^a, will be skewed with $X^{[b} \ldots Z^{d]}$ and will thus give zero. Thus (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1). Hence (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3).

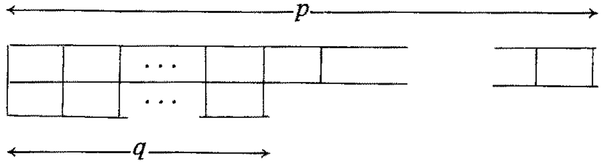
Consider a function $f(\xi^a, \eta^b)$, homogeneous of degrees q and p in ξ^a and η^b, respectively, such that

$$f(\xi^a + \lambda\eta^a, \eta^b) = f(\xi^a, \eta^b)$$

In the particular case when $f(\xi^a, \eta^b)$ can be written as a polynomial

$$f(\xi^a, \eta^b) = f_{a\cdots c\, e\cdots g}\,\xi^a \cdots \xi^c \eta^e \cdots \eta^g$$

where $f_{a\cdots c\, e\cdots g} = f_{(a\cdots c)\,(e\cdots g)}$ and $f_{a\cdots(ce\cdots g)} = 0$. All such $f_{a\cdots ce\cdots g}$'s form an irreducible representation space of the Lie algebra of $GL(n, C)$, if $f(\xi^a, \eta^b)$ is singular outside some domain (inside which it is nonsingular). The $f_{a\cdots ce\cdots g}$'s can be represented by the Young tableau

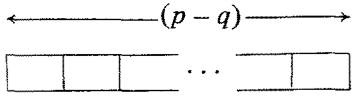


If we are considering $GL(2, C)$, ξ^a and η^b are two-component complex vectors. In that case $f(\xi^a, \eta^b)$ satisfies the conditions for Theorem 1. Thus we can write

$$f(\xi^a, \eta^b) = f_1(X, \eta^b) = X^q g(\eta^b) \tag{2.5}$$

where $g(\eta^b)$ is homogeneous of degree $(p - q)$ in η^b, $X = \epsilon_{ab}\xi^a\eta^b$. This method will be used often later.

If one is dealing with $SL(2, C)$ then $X = 1$. In terms of Young tableau, the previous Young tableau would become



This can be generalized for $SL(n, C)$.

The functions used for getting representations of the Lie algebra of $GL(n, C)$ satisfy the condition that

$$f(W^a + \lambda X^a, X^b, \ldots, Z^d) = \cdots = f(W^a + \mu Z^a, X^b, \ldots, Z^d) = f(W^a, X^b + \nu Y^b, \ldots, Z^d) = f(W^a, X^b + \rho Z^b, \ldots, Z^d) = \cdots = f(W^a, X^b, \ldots, Z^d)$$

and they are homogeneous in $W^a, X^b, \ldots, Z^d$ of degrees $p, q, \ldots, s$. If they can be expressed as polynomials, i.e.,

$$f(W^a, X^b, \ldots, Z^d) = f_{a\cdots c\, e\cdots g\, k\cdots m\, n\cdots p} W^a \cdots W^c X^e \cdots X^g \times Y^k \cdots Y^m Z^n \cdots Z^p \tag{2.6}$$

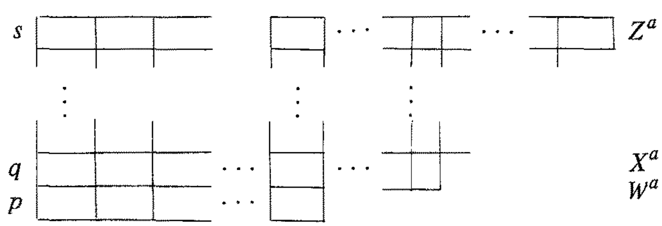
where

$$f_{a\cdots c\, e\cdots g\, k\cdots m\, n\cdots p} = f_{(a\cdots c)(e\cdots g)(k\cdots m)(n\cdots p)}$$

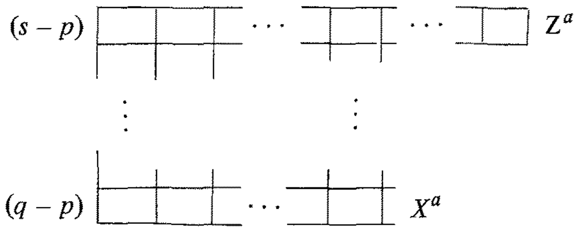
and

$$f_{a\cdots c\, e\cdots g\, k\cdots (mn\cdots p)} = f_{a\cdots c\, e\cdots (gk\cdots m) n\cdots p}$$
$$= f_{a\cdots (ce\cdots g)\, k\cdots m\, n\cdots p} = f_{a\cdots c\, e\cdots (g|k\cdots m|n\cdots p)}$$
$$= f_{a\cdots (c|e\cdots g\, k\cdots m|n\cdots p)} = f_{a\cdots (c|e\cdots g|k\cdots m) n\cdots p} = 0$$

All such $f_{a\cdots c\, e\cdots g\, k\cdots m\, n\cdots p}$'s span an irreducible representation of the Lie algebra of $GL(n, C)$. They can be expressed by the Young tableau



If one deals with $SL(n, C)$ the Young tableau becomes



In terms of the functions we have

$$f(W^a, X^b, \ldots, Z^d) = \gamma^p g(X^b, \ldots, Z^d) \tag{2.7}$$

and for $SL(n, C)$, $\gamma = 1$, γ being $\epsilon_{ab\cdots d} W^a X^b \cdots Z^d$.

For convenience I shall use the notation $\eta^{a\alpha}$ for the set of vectors $(W^a, X^a, \ldots, Z^a)$, α running from 1 to n as does a. Thus η^{a1} is W^a, η^{a2} is X^a and so on until η^{an} is Z^a. The equation (2.4) is equivalent to

$$f(\eta^{a\alpha}) = f(k^a_b \eta^{b\alpha}) \tag{2.8}$$

where $k^a_b = 0$ for $a < b$, $= 1$ for $a = b$ and has some sufficiently small value for $a > b$. For such f's it is easy to see that

$$\begin{aligned} \eta^{a\alpha}\partial_{a\beta} f(\eta^{c\gamma}) &= 0 \qquad \text{for } \alpha > \beta \\ &= h_\beta f(\eta^{c\gamma}) \qquad \text{for } \alpha = \beta \end{aligned} \tag{2.9}$$

where $\partial_{a\alpha} = \partial/\partial\eta^{a\alpha}$, i.e., $\partial_{a0} = \partial/\partial W^a$, etc., up to $\partial_{an} = \partial/\partial Z^a$. We can write γ as

$$\gamma = \epsilon_{a\cdots c}\epsilon_{\alpha\cdots\gamma}\eta^{a\alpha}\cdots\eta^{c\gamma}/n! \tag{2.10}$$

I shall now define the Casimir operators and then evaluate them for $GL(2, C)$, $GL(3, C)$ and $GL(4, C)$.

3. *Casimir Operators*

The canonical generators are defined by

$$K^b_a = \partial_{a\alpha} \circ \eta^{b\alpha} = \eta^{b\alpha}\partial_{a\alpha} + n\delta^b_a \tag{3.1}$$

All contractions of these will give Casimir operators. Thus

$$K_1 = K^a_a, \qquad K_2 = K^b_a K^a_b, \qquad K_3 = K^b_a K^c_b K^a_c, \qquad \text{etc.}$$

are Casimir operators. The set $\{K_i\}$ $(i = 1, \ldots, n)$ are linearly independent Casimir operators, all other Casimir operators may be expressed in terms of these.

It is easily seen that the functions defined by (2.9) are eigenfunctions of the Casimir operators by commuting the appropriate $\eta^{a\alpha}$'s past the $\partial_{b\beta}$'s [so that (2.9) can be used] and adding in the extra term. In fact it is for this reason that the representation was chosen to be of the form satisfying (2.9).

Consider the case of $GL(2, C)$. Here we have only the Casimir operators to work out:

$$K_1 f(\eta^{b\beta}) = (\eta^{a\alpha}\partial_{a\alpha} + 2\delta_a^a) f(\eta^{b\beta}) = \left(\sum_{a=1}^{2} h_a + 4\right) f(\eta^{b\beta})$$
$$= (h_1 + h_2 + 4) f(\eta^{b\beta}) \tag{3.2}$$
$$K_2 f(\eta^{c\gamma}) = (\partial_{a\alpha}\eta^{b\alpha}\partial_{b\beta}\eta^{a\beta}) f(\eta^{c\gamma}) = (\delta_\beta^\beta \partial_{a\alpha}\eta^{a\alpha} + \partial_{a\alpha}\eta^{b\alpha}\eta^{a\beta}\partial_{b\beta}) f(\eta^{c\gamma})$$

Using (2.9) we can write

$$K_2 f(\eta^{c\gamma}) = (2K_1 + h_\alpha \partial_{a\alpha}\eta^{a\alpha} + \partial_{ax}\eta^{bx}\eta^{ay}\partial_{by}) f(\eta^{c\gamma})$$

where $x < y$ and where $h_\alpha \partial_{a\alpha}\eta^{a\alpha}$ stands for $(\partial_{a1}\eta^{a1})h_1 + \cdots + (\partial_{an}\eta^{an})h_n$. Now we have

$$(\partial_{ax}\eta^{bx}\eta^{ay}\partial_{by}) f(\eta^{c\gamma}) = \left(\sum_{x<y} \eta^{ay}\partial_{ay} + \eta^{bx}\eta^{ay}\partial_{by}\partial_{ax}\right) f(\eta^{c\gamma}), \qquad x < y$$

since $\partial_{ax}\partial_{by} = \partial_{by}\partial_{ax}$. Using (2.9) again, with $x < y$, we have

$$(\eta^{bx}\eta^{ay}\partial_{by}\partial_{ax}) f(\eta^{c\gamma}) = (\eta^{bx}\partial_{by}\eta^{ay}\partial_{ax} - \sum_{y>x} \eta^{ax}\partial_{ax}) f(\eta^{c\gamma})$$

I shall write

$$\sum_{x<y=1}^{n}$$

for

$$\sum_{y=x+1}^{n} \; \sum_{x=1}^{n-1}$$

Thus we have

$$K_2 f(\eta^{c\gamma}) = \left(2K_1 + \delta_a^a \sum_{\alpha=1}^{2} h_\alpha + \sum_{\alpha=1}^{2} h_\alpha^2 + \sum_{x<y=1}^{2} (h_y - h_x)\right) f(\eta^{c\gamma})$$
$$= (8 + 3h_1 + 5h_2 + h_1^2 + h_2^2) f(\eta^{c\gamma}) \tag{3.3}$$

The Casimir operators for $GL(3, C)$ and $GL(4, C)$ are worked out and given in the Appendix. The expressions for the Casimir operators K_q rapidly become too cumbersome for higher values of n and larger values of q. The situation becomes much more manageable if we replace the degrees of homogeneity by what I shall call "numbers of homogeneity" defined by

$$N_i = h_i - i + 1 \tag{3.4}$$

Now for $GL(2, C)$ we get

$$\begin{aligned} {}^2K_1 f &= [3 + (N_1 + N_2)] f \\ {}^2K_2 f &= [4 + 3(N_1 + N_2) + N_1^2 + N_2^2] f \end{aligned} \tag{3.5}$$

For $GL(3, C)$ we get

$$\begin{aligned}
{}^3K_1 f &= [6 + (N_1 + N_2 + N_3)]f \\
{}^3K_2 f &= [10 + 4(N_1 + N_2 + N_3)]f \\
{}^3K_3 f &= [15 + 10(N_1 + N_2 + N_3) + 5(N_1^2 + N_2^2 + N_3^2) \\
&\quad + (N_1N_2 + N_2N_3 + N_3N_1) + (N_1^3 + N_2^3 + N_3^3)]f \qquad (3.6)
\end{aligned}$$

For $GL(4, C)$ we get

$$\begin{aligned}
{}^4K_1 f &= [10 + (N_1 + N_2 + N_3 + N_4)]f \\
{}^4K_2 f &= [20 + 5(N_1 + N_2 + N_3 + N_4) + (N_1^2 + N_2^2 + N_3^2 + N_4^2)]f \\
{}^4K_3 f &= \{35 + 15(N_1 + N_2 + N_3 + N_4) + 6(N_1^2 + N_2^2 + N_3^2 + N_4^2) \\
&\quad + [N_1(N_2 + N_3 + N_4) + N_2(N_3 + N_4) + N_3N_4] \\
&\quad + (N_1^3 + N_2^3 + N_3^3 + N_4^3)\}f \\
{}^4K_4 f &= \{56 + 35(N_1 + N_2 + N_3 + N_4) + 21(N_1^3 + N_2^3 + N_3^3 + N_4^3) \\
&\quad + 6[N_1(N_2 + N_3 + N_4) + N_2(N_3 + N_4) + N_3N_4] \\
&\quad + [N_1^2(N_2 + N_3 + N_4) + N_2^2(N_3 + N_4) + N_3^2N_4] \\
&\quad + [N_1(N_2^2 + N_3^2 + N_4^2) \\
&\quad + N_2(N_3^2 + N_4^2) + N_3N_4^2] + 7(N_1^3 + N_2^3 + N_3^3 + N_4^3) \\
&\quad + (N_1^4 + N_2^4 + N_3^4 + N_4^4)\}f \qquad (3.7)
\end{aligned}$$

The eigenvalue expression of the Casimir operators can be still more simply expressed as symmetric combinations of the numbers of homogeneity:

$$S_1 = \sum_{i=1}^{n} N_i, \qquad S_2 = \sum_{i=1}^{n} N_i^2, \qquad \text{etc.}$$

$$\begin{aligned}
{}^2K_1 f &= (3 + S_1)f \\
{}^2K_2 f &= (4 + 3S_1 + S_2)f \\
{}^3K_1 f &= (6 + S_1)f \\
{}^3K_2 f &= (10 + 4S_1 + S_2)f \\
{}^3K_3 f &= (15 + 10S_1 + \tfrac{1}{2}S_1^2 + 4\tfrac{1}{2}S_2 + S_3)f \\
{}^4K_1 f &= (10 + S_1)f \\
{}^4K_2 f &= (20 + 5S_1 + S_2)f \\
{}^4K_3 f &= (35 + 15S_1 + \tfrac{1}{2}S_1^2 + 5\tfrac{1}{2}S_2 + S_3)f \\
{}^4K_4 f &= (36 + 35S_1 + 3S_1^2 + \tfrac{1}{3}S_1^3 + 18S_2 + 6\tfrac{2}{3}S_3 + S_4)f \qquad (3.8)
\end{aligned}$$

To appreciate the simplicity of these results, compare the ${}^4K_q$ with the corresponding values for $U(2, 2)$ given by Tsu Yao (1967, 1968). I shall now consider the effect of restricting to $SL(n, C)$ from $GL(n, C)$.

4. *Restriction to SL(n, C)*

The formalism given earlier can be used to get the Casimir operators when the group is unimodular instead of general, i.e., for $SL(n, C)$ instead of $GL(n, C)$, by making the canonical generators traceless and defining the Casimir operators as contractions of the new generators. Calling the canonical generators $(SK)_a^b$, we have

$$(SK)_a^b = K_a^b - (1/n)K_c^c\delta_a^b \tag{4.1}$$

$$(SK_1) = (SK)_a^a = 0 \tag{4.2}$$

As an example, consider the case of $n = 4$.

$$(SK)_2 = (K_a^b - \tfrac{1}{4}K_1\delta_b^a)(K_b^a - \tfrac{1}{4}K_1\delta_b^a) = K_2 - \tfrac{1}{4}K_1^2 \tag{4.3}$$

$$(SK)_3 = (K_a^b - \tfrac{1}{4}K_1\delta_a^b)(K_b^c - \tfrac{1}{4}K_1\delta_b^c)(K_c^a - \tfrac{1}{4}\delta_c^a) = K_3 - \tfrac{3}{4}K_2K_1 + \tfrac{1}{2}K_1^3 \tag{4.4}$$

$$\begin{aligned}(SK)_4 &= (K_a^b - \tfrac{1}{4}K_1\delta_a^b)(K_b^c - \tfrac{1}{4}K_1\delta_b^c)(K_c^d - \tfrac{1}{4}\delta_c^d)(K_d^a - \tfrac{1}{4}\delta_d^a)\\ &= K_4 - K_3K_1 + \tfrac{3}{2}K_2K_1^2\end{aligned} \tag{4.5}$$

The reader will have noticed that the Casimir operators of the unimodular case are quite complicated.

5. *The Symmetry Properties of the Casimir Operators*

In Section 3 the Casimir operators of $GL(2, C)$, $GL(3, C)$, and $GL(4, C)$ were shown to be symmetrical in the numbers of homogeneity of the eigenfunctions of the operators. The proof that the Casimir operators of $GL(n, C)$ are symmetric in the numbers of homogeneity (which will be presented now) depends on the functions being singular somewhere, but having a domain over which they are nonsingular.

Consider a function

$$f(\eta^{a\alpha}) = f(W^a, \ldots, Z^a) \tag{5.1}$$

such that

$$\begin{aligned} f(\eta^{a\alpha}) = f(\eta^{1a\alpha}) \text{ where } \eta^{1a\alpha} &= (k_\beta^\alpha + \delta_\beta^\alpha)\eta^{a\beta}\\ k_\beta^\alpha = 0 \qquad &\text{if } \alpha \geqslant \beta\\ &= \text{some small constant if } \alpha < \beta\end{aligned} \tag{5.2}$$

Theorem 1 can be used because of (5.1):

$$f(W^a, \ldots, Z^d) = g(W^{[a}, \ldots, Z^{d]}, X^p, \ldots, Z^a) \tag{5.3}$$

where f is homogeneous of degrees $h_1, \ldots, h_n$ in $W^a, \ldots, Z^d$, respectively. Hence we can write

$$f(W^a, \ldots, Z^d) = \gamma^{h_1} F_1(X^p, \ldots, Z^r) \tag{5.4}$$

where $\gamma = \epsilon_{a \cdots d} W^{[a} \cdots Z^{d]}$ where F_1 is homogeneous of degrees $h_2 - h_1, \ldots, h_n - h_1$ in $X^p, \ldots, Z^r$, respectively. Consider an $F_2(X^p, \ldots, Z^r)$ with the same degrees of homogeneity as $F_1(X^p, \ldots, Z^r)$ and with appropriate singularities around which one can perform contour integrals. We can then define [as was done for $SL(2, C)$ by R. Penrose (1967, 1968, 1973)]

$$F(W^a, \ldots, Z^p) = \gamma^{h_1} \oint F_2(W^p + \lambda_2 Z^p, \ldots, Y^r + \lambda_n Z^r)\, d\lambda_2 \wedge \ldots \wedge d\lambda_n \tag{5.5}$$

This can be written as

$$F(\eta^{a\alpha}) = \oint f(\eta^{1a\hat{\alpha}})\, d\lambda_2 \wedge \ldots \wedge d\lambda_n \tag{5.6}$$

where $\eta^{1a\hat{\alpha}} = \Lambda_\alpha^{1\hat{\alpha}} \eta^{a\alpha}$;

$$\Lambda_\alpha^{1\hat{\alpha}} = \begin{pmatrix} 0 & 0 & 0 \cdots 0 & 1 \\ 1 & 0 & 0 \cdots 0 & \lambda_2 \\ 0 & 1 & 0 \cdots 0 & \lambda_3 \\ 0 & 0 & 0 \cdots 1 & \lambda_n \end{pmatrix} \tag{5.7}$$

Clearly K_a^b is not affected by transforming $\eta^{a\alpha}$ by $\Lambda_\alpha^{1\hat{\alpha}}$ as we shall have $\partial_{1a\hat{\alpha}} = V_{1\alpha}^{\alpha} \partial_{a\hat{\alpha}}$ where $V_{1\alpha}^{\alpha} \Lambda_\beta^{1\alpha} = \delta_\beta^\alpha$. Thus

$$K_a^b = \partial_{a\alpha} \eta^{b\alpha} = \partial_{1a} \eta^{1a\hat{\alpha}} = \hat{K}_a^b \tag{5.8}$$

Similarly we define

$$G(W^a, \ldots, Z^a) = \gamma^{h_1} \oint F_2(Z^a, X^b + \lambda_3 Z^b, \ldots, Y^\alpha + \lambda_n Z^d)\, d\lambda_3 \wedge \ldots \wedge d\lambda_n \tag{5.9}$$

which can be written as

$$G(\eta^{a\alpha}) = \oint f(\eta^{2a\hat{\alpha}})\, d\lambda_3 \wedge \ldots \wedge d\lambda_n \tag{5.10}$$

where $\eta^{2a\hat{\alpha}} = \Lambda_\alpha^{2\hat{\alpha}} \eta^{a\alpha}$

$$\Lambda_\alpha^{2\hat{\alpha}} = \begin{pmatrix} 1 & 0 & 0 \cdots 0 & 0 \\ 0 & 0 & 0 \cdots 0 & 1 \\ 0 & 1 & 0 \cdots 0 & \lambda_3 \\ 0 & 0 & 0 \cdots 1 & \lambda_n \end{pmatrix} \tag{5.11}$$

As before, K_a^b is not affected by transforming $\eta^{a\alpha}$ by $\Lambda_\alpha^{2\hat{\alpha}}$. We have the relationships

$$\Lambda_\alpha^{i\hat{\alpha}} V_{j\hat{\beta}}^{\alpha} = \delta_{\hat{\beta}}^{\hat{\alpha}} \delta_j^i; \qquad \Lambda_\alpha^{i\hat{\alpha}} V_{j\hat{\alpha}}^{\beta} = \delta_\alpha^\beta \delta_j^i \qquad (i,j=1,2) \tag{5.12}$$

I shall now show that the numbers of homogeneity of F and G are permutations of the numbers of homogeneity of f and that they satisfy the condition (2.8). I shall then prove that F and G have the same Casimir operators as f and hence that K is symmetric in the numbers of homogeneity

$$F(W^a, \ldots, AZ^d) = \gamma^{h_1} A^{h_1} \oint F_2(W^p + A\lambda_2 Z^p, \ldots, Y^r + A\lambda_n Z^r)\, d\lambda_2 \wedge \ldots \wedge d\lambda_n$$

Change the variables λ_i to $\lambda_i' = A\lambda_i$. Then

$$F(W^a, \ldots, AZ^d) = \lambda^{h_1} A^{h_1 - n + 1} \oint F_2(W^p + \lambda_2 Z^p, \ldots, Y^r + \lambda_n' Z^r)\, d\lambda_2' \wedge \ldots \wedge d\lambda_n' = A^{h_1 - n+1} F(W^a, \ldots, Z^d) \tag{5.13}$$

$$F(W^a, \ldots, BY^c, BZ^d) = \gamma^{h_1} B^{2h_1} \oint F_2(W^p + B\lambda_2 Z^p, \ldots, \times B[Y^r + \lambda_n Z^r])\, d\lambda_2 \cdots d\lambda_n$$

Now F_2 is homogeneous of degree $h_n - h_1$ in $Y^r + \lambda_n Z^r$. Since there are now only $n-2$, λ's to be changed to λ's,

$$\begin{aligned} F(W^a, \ldots, BY^c, BZ^d) &= B^{h_n + h_1 - n + 2} F(W^a, \ldots, Y_c, Z^d) \\ F(W^a, \ldots, BY^c, Z^d) &= B^{h_n + 1} F(W^a, \ldots, Y^c, Z^d) \end{aligned} \tag{5.14}$$

Similarly, we can continue to

$$F(W^a, DX^b, \ldots, Z^d) = D^{h_3+1} F(W^a, X^b, \ldots, Z^d) \tag{5.15}$$

$$F(EW^a, X^b, \ldots, Z^d) = E^{h_2+1} F(W^a, X^b, \ldots, Z^d) \tag{5.16}$$

Thus, if the numbers of homogeneity of f are $(N_1, \ldots, N_n)$, the numbers of homogeneity of F are $(N_2, \ldots, N_n, N_1)$, i.e., cyclic permutations of $N_1, \ldots, N_n$.

$$G(W^a, \ldots, AZ^d) = \gamma^{h_1} A^{h_1} \oint F_2(AZ^a, X^b + \lambda_3 Z^b, \ldots, Y^d + \lambda_n Z^d)\, d\lambda_3 \wedge \ldots \wedge d\lambda_n = A^{h_2 - n + 2} G(W^a, \ldots, Z^d) \tag{5.17}$$

$$G(W^a, \ldots, BY^c, Z^d) = B^{h_n+1} G(W^a, \ldots, Y^c, Z^d) \tag{5.18}$$

$$G(W^a, DX, \ldots, Z^d) = D^{h_3+1} G(W^a, X^b, \ldots, Z^d) \tag{5.19}$$

$$G(EW^a, X^b, \ldots, Z^d) = E^{h_1} G(W^a, X^b, \ldots, Z^d) \tag{5.20}$$

Thus the numbers of homogeneity of G are $(N_1, N_3, \ldots, N_n, N_2)$, i.e., leaving N_1 alone, cyclic permutations of $N_2, \ldots, N_n$. One could similarly start with functions having the same homogeneity degrees as F or G and get further permutations of the N_i's:

$$F(W^a, \ldots, Y^c + kZ^c, Z^d) = \gamma^{h_1} \oint F_2(W^p + \lambda_2 Z^p, \ldots, Y^r + kZ^r + \lambda_n Z^r)\, d\lambda_2 \wedge \ldots \wedge d\lambda_n$$

Defining a new $\lambda_n' = \lambda_n + k$, $d\lambda_n' = d\lambda_n$. Thus

$$F(W^a, \ldots, Y^c + kZ^c, Z^d) = F(W^a, \ldots, Y^c, Z^d) \tag{5.21}$$

To see how this works for other combinations it is convenient to consider, as an example, $GL(4, C)$. Then from

$$F(W^a, X^b + kZ^d, Y^c, Z^d) = F(W^a + kZ^a, X^b, Y^c, Z^d) = F(W^a, X^b, Y^c, Z^d) \tag{5.22}$$

follows as before:

$$F(W^a, X^b + kY^b, Y^c, Z^d) = \gamma^{h_1} \oint F_2(W^a + \lambda_2 Z^a, X^b + kY^b + \lambda_3 Z^b, Y^c + \lambda_4 Z^c)\, d\lambda_2 \wedge d\lambda_3 \wedge d\lambda_4$$

Using (5.21) we get

$$F(W^a, X^b + kY^b, Y^c, Z^d) = \gamma^{h_1} \oint F_2(W^a + \lambda_2 Z^a, X^b + (\lambda_3 - k\lambda_4)Z^b, Y^c + \lambda_4 Z^c) d\lambda_2 \wedge d\lambda_3 \wedge d\lambda_4$$

Defining $\lambda_3' = \lambda_3 - k\lambda_4$, we get $d\lambda_3' = d\lambda_3 - kd\lambda_4$. Now we have $d\lambda_4 \wedge d\lambda_4 = 0$, hence $d\lambda_3' \wedge d\lambda_4 = d\lambda_3 \wedge d\lambda_4$. Thus

$$F(W^a, X^b + kY^b, Y^c, Z^d) = F(W^a, X^b, Y^c, Z^d) \tag{5.23}$$

Similarly

$$F(W^a + kY^a, X^b, Y^c, Z^d) = F(W^a, X^b, Y^c, Z^d) \tag{5.24}$$

$$F(W^a + kX^a, X^b, Y^c, Z^d) = \gamma^{h_1} \oint F_2(W^a + kX^a + \lambda_2 Z^a, X^b + \lambda_3 Z^b, Y^c + \lambda_4 Z^c)\, d\lambda_2 \wedge d\lambda_3 \wedge d\lambda_4 = \gamma^{h_1} \oint F_2(W^a + (\lambda_2 - k\lambda_3)Z^a, X^b + \lambda_3 Z^b, Y^c + \lambda_4 Z^c)\, d\lambda_2 \wedge d\lambda_3 \wedge d\lambda_4$$

Defining $\lambda_2' = \lambda_2 - k\lambda_3$, it is clear that we get

$$F(W^a + kX^a, X^b, Y^c, Z^d) = F(W^a, X^b, Y^c, Z^d) \tag{5.25}$$

Similarly for $G(W^a, X^b, Y^c, Z^d)$ in $GL(4, C)$ we get

$$G(W^a + kZ^a, X^b, Y^c, Z^d) = G(W^a + kY^a, X^b, Y^c, Z^d) = G(W^a + kX^a, X^b, Y^c, Z^d) = G(W^a, X^b, Y^c, Z^d) \tag{5.26}$$

as we have W^a appearing only in γ^{h_1} and hence skewed with X^b, Y^c and Z^d. Also

$$G(W^a, X^b, Y^c + kZ^c, Z^d) = G(W^a, X^b + kZ^b, Y^c, Z^d) = G(W^a, X^b, Y^c, Z^d) \tag{5.27}$$

as in the proof of (5.21).

$$G(W^a, X^b + kY^b, Y^c, Z^d) = G(W^a, X^b, Y^c, Z^d) \tag{5.28}$$

as in the proof of (5.23).

Hence

> *Theorem 2:* If $f(n^{a\alpha})$ are homogeneous functions satisfying (2.8), the functions
>
> $$F(\eta^{a\alpha}) = \oint f(\eta^{1a\alpha})\, d\lambda_2 \wedge \dots \wedge d\lambda_n$$
> $$G(\eta^{a\alpha}) = \oint f(\eta^{2a\alpha})\, d\lambda_3 \wedge \dots \wedge d\lambda_n$$
>
> satisfy (2.8), and their numbers of homogeneity are permutations of the numbers of homogeneity of $f(\eta^{a\alpha})$.

Since K_a^b is unaffected by the transformations $\eta^{a\alpha} \to \eta^{ia\alpha}$ $(i = 1, 2)$, we see that the eigenvalue of F is the same as the eigenvalue of f. Similarly for G.

> *Theorem 3:* ${}^nK_q$ are symmetric in the numbers of homogeneity for all values of n and q.

Writing down the most general form for the eigenvalues of ${}^nK_q$ in terms of numbers of homogeneity of its eigenfunctions, so that it is symmetric in them, we have

$$ {}^nK_q f(\eta^{a\alpha}) = \left(\sum_{j=i}^{n} \sum_{p=0}^{q} K_{q(p)} N_j^p + \sum_{i=2}^{q-1} \sum_{r_1 r_2 \cdots r_i = 1 \cdots p_j} \sum_{\Sigma p < q - i + 1} K_{q(p \cdots p_i)} N_{r_i}^{p_1} \cdots N_{r_i}^{p_i} \right) f(\eta^{a\alpha}) \tag{5.29}$$

where $K_{q(p)}$ and $K_{q(p_1 \dots p_i)}$ are combinatorial factors given by

$$K_{q(p_1 \dots p_i)} = \binom{n + q - 1}{q - s_i} \tag{5.30}$$

where

$$s_i = \sum_{j=1}^{i} p_j, \; K_{q(p_1)} \equiv K_{q(p)}$$

This may be seen to follow from the number of ways the canonical generators can be commuted.

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Appendix

From now on I shall write f instead of $f(\eta^{c\gamma})$. Also, to simplify the calculations, I shall put terms that are already calculated (or may easily be calculated) into $\langle\,\rangle$ brackets and ignore them until the end of the calculation. I shall now work out the Casimir operator of $GL(3, C)$:

$$K_1 f = (\partial_{a\alpha}\eta^{a\alpha})f = (\delta^a_a\delta^\alpha_\alpha + \eta^{a\alpha}\partial_{a\alpha})f$$

$$= \left(3 \times 3 + \sum_{\alpha=1}^{3} h_\alpha\right) f = (9 + h_1 + h_2 + h_3)f \tag{A1}$$

$$K_2 f = (\partial_{a\alpha}\eta^{b\alpha}\partial_{b\beta}\eta^{a\beta})f = (\delta^\beta_\beta\partial_{a\alpha}\eta^{a\alpha} + \partial_{a\alpha}\eta^{b\alpha}\eta^{a\beta}\partial_{b\beta})f$$

$$= (\langle 3K_1\rangle + h_\alpha\partial_{a\alpha}\eta^{a\alpha} + \partial_{ax}\eta^{bx}\eta^{ay}\partial_{by})f \qquad (x < y)$$

$$= \left(\left\langle \sum_{\alpha=1}^{3} h_\alpha(3 + h_\alpha)\right\rangle + \left\langle \sum_{x<y=1}^{3} (h_y - h_x)\right\rangle\right) f$$

$$K_2 f = (27 + 4h_1 + 6h_2 + 8h_3 + h_1^2 + h_2^2 + h_3^2)f \tag{A2}$$

$$K_3 f = (K^b_a K^c_b \partial_{c\gamma}\eta^{a\gamma})f = (\langle 3K_2\rangle + K^b_a\partial_{b\beta}\eta^{c\beta}\eta^{a\gamma}\partial_{c\gamma})f$$

$$= (h_\beta K^b_a\partial_{b\beta}\eta^{a\beta} + K^b_a\partial_{bx}\eta^{cx}\eta^{ay}\partial_{cy})f \qquad (x < y)$$

$$= \left(\left\langle \sum_{\beta=1}^{3} h_\beta K_1\right\rangle + h_\beta\partial_{a\alpha}\eta^{b\alpha}\eta^{a\beta}\partial_{b\beta} + \langle A\rangle\right) f$$

$$= (h_\alpha^2\partial_{a\alpha}\eta^{a\alpha} + h_y\partial_{ax}\eta^{bx}\eta^{ay}\partial_{by})f \qquad (x < y)$$

$$= \left(\left\langle \sum_{\alpha=1}^{3} h_\alpha^2(3 + h_\alpha)\right\rangle + \left\langle \sum_{x<y=1}^{3} h_y(h_y - h_x)\right\rangle\right) f$$

$$Af = (\partial_{a\alpha}\eta^{b\alpha}\partial_{bx}\eta^{cx}\eta^{ay}\partial_{cy})f$$

$$= (\partial_{a\alpha}\eta^{b\alpha}\eta^{cx}\eta^{ay}\partial_{cy}\partial_{bx} + \sum_{x<y} \partial_{a\alpha}\eta^{b\alpha}\eta^{ay}\partial_{by})f$$

$$= (\partial_{a\alpha}\eta^{cx}\eta^{ay}\partial_{cy}\eta^{b\alpha}\partial_{bx} - \partial_{ay}\eta^{bx}\eta^{ay}\partial_{bx}$$

$$+ \sum_{x<y} h_y\partial_{ay}\eta^{ay} + \sum_{x<y} \partial_{aw}\eta^{bw}\eta^{ay}\partial_{by})f \qquad (w < y)$$

$$= \Big[h_x\eta_{ax}\eta^{cx}\eta^{ay}\partial_{cy} + \partial_{av}\eta^{cx}\eta^{ay}\partial_{cy}\eta^{bv}\partial_{bx}$$

$$- \Big\langle \sum_{x<y=1}^{3} h_x(h_y+3)\Big\rangle + \Big\langle \sum_{x<y=1}^{3} h_y(3+h_y)\Big\rangle$$

$$+ \Big\langle \sum_{x,w\,y=1}^{3} (h_y - h_w)\Big\rangle\Big] f \qquad (v<x<y)$$

$$= \Big[\Big\langle \sum_{x<y=1}^{3} (h_x h_y - h_x)\Big\rangle + \Big\langle \sum_{v<x<y=1}^{3} (h_v - h_x)\Big\rangle\Big] f$$

$$K_3 f = (81 + 13h_1 + 25h_2 + 43h_3 + 5h_1^2 + 8h_2^2 + 11h_3^2 + h_1h_2 + h_2h_3 + h_3h_1 + h_1^2 + h_2^2 + h_3^2)f \qquad \text{(A3)}$$

I shall now work out the Casimir operators of $GL(4, C)$ (cf. Qadir, 1971):

$$K_1 f = (\partial_{a\alpha}\eta^{a\alpha})f = (\delta_a^a\delta_\alpha^\alpha + \eta^{a\alpha}\partial_{a\alpha})f = \Big(4 \times 4 + \sum_{\alpha=1}^{4} h_a\Big)$$

$$= (16 + h_1 + h_2 + h_3 + h_4)f \qquad \text{(A4)}$$

$$K_2 f = (K_a^b\partial_{b\beta}\eta^{a\beta})f = (\langle 4K_1\rangle + \partial_{a\alpha}\eta^{b\alpha}\eta^{a\beta}\partial_{b\beta})f$$

$$= (h_\alpha\partial_{a\alpha}\eta^{a\alpha} + \partial_{ax}\eta^{bx}\eta^{ay}\partial_{by})f$$

$$= \Big(\Big\langle \sum_{\alpha=1}^{4} h_\alpha(4+h_\alpha)\Big\rangle + \Big\langle \sum_{x<y=1}^{4} (h_y - h_x)\Big\rangle\Big) f$$

$$K_2 f = (64 + 5h_1 + 7h_2 + 9h_3 + 11h_4 + h_1^2 + h_2^2 + h_3^2 + h_4^2)f \qquad \text{(A5)}$$

$$K_3 f = (K_a^b K_b^c \partial_{c\gamma}\eta^{a\gamma})f = (\langle 4K_2\rangle + K_a^b\partial_{b\beta}\eta^{c\beta}\eta^{a\gamma}\partial_{c\gamma})f$$

$$= (h_\beta K_a^b\partial_{b\beta}\eta^{a\beta} + K_a^b\partial_{bx}\eta^{cx}\eta^{a}\partial_{cy})f$$

$$= \Big(\Big\langle \sum_{\beta=1}^{4} h_\beta K_1\Big\rangle + h_\beta\partial_{a\alpha}\eta^{b\alpha}\eta^{a\beta}\partial_{b\beta} + \langle A\rangle\Big) f$$

$$= (h_\alpha^2\partial_{a\alpha}\eta^{a\alpha} + h_y\partial_{ax}\eta^{bx}\eta^{ay}\partial_{by})f$$

$$= \Big[\Big\langle \sum_{\alpha=1}^{4} h_\alpha^2(4+h_\alpha)\Big\rangle + \Big\langle \sum_{x<y=1}^{4} h_y(_y - h_x)\Big\rangle\Big] f$$

$$Af = (\partial_{a\alpha}\eta^{b\alpha}\eta^{cx}\eta^{ay}\partial_{cy}\partial_{bx} + \sum_{x<y} \partial_{a\alpha}\eta^{b\alpha}\eta^{ay}\partial_{by})f$$

$$= (\partial_{a\alpha}\eta^{cx}\eta^{ay}\partial_{cy}\eta^{b\alpha}\partial_{bx} - \partial_{ay}\eta^{by}\eta^{ay}\partial_{bx}$$

$$+\sum_{x<y} h_y\partial_{ay}\eta^{ay}+\sum_{x<y}\partial_{aw}\eta^{bw}\eta^{ay}\partial_{by})f$$

$$=(h_x\partial_{ax}\eta^{cx}\eta^{ay}\partial_{cy}+\partial_{av}\eta^{cx}\eta^{ay}\partial_{cy}\eta^{bv}\partial_{bx})f$$

$$-\left[\left\langle\sum_{x<y=1}^{4} h_x(h_y+4)\right\rangle+\left\langle\sum_{x<y=1}^{4} h_y(4+h_y)\right\rangle\right.$$

$$\left.+\left\langle\sum_{w,x<y=1}^{4}(h_y-h_w)\right\rangle\right]f$$

$$=\left[\left\langle\sum_{x<y=1}^{4} h_x(h_y-h_x)\right\rangle+\left\langle\sum_{v<x<y=1}^{4}(h_v-h_x)\right\rangle\right]f$$

$$K_3f=(256+21h_1+35h_2+55h_3+81h_4+h_1(h_2+h_3+h_4)+h_2(h_3+h_4)$$

$$+h_3h_4+6h_1^2+9h_2^2+12h_3^2+15h_4^2+h_1^3+h_2^3+h_3^3+h_4^3)f \qquad \text{(A6)}$$

$$K_4f=(K_a^bK_b^cK_c^d\partial_{d\delta}\eta^{a\delta})f=(\langle 4K_3\rangle+K_a^bK_b^c\partial_{c\gamma}\eta^{d\gamma}\eta^{a\delta}\partial_{d\delta})f$$

$$=(h_\gamma K_a^bK_b^c\partial_{c\gamma}\eta^{a\gamma}+K_a^bK_b^c\partial_{cy}\eta^{dy}\eta^{az}\partial_{dz})f$$

$$=\left(\left\langle\sum_{\gamma=1}^{4} h_\gamma K_2\right\rangle+h_\gamma K_a^b\partial_{b\beta}\eta^{c\beta}\eta^{a\gamma}\partial_{c\gamma}+\langle B\rangle\right)f$$

$$=(h_\beta^2K_a^b\partial_{b\beta}\eta^{a\beta}+h_yK_a^b\partial_{bx}\eta^{cx}\eta^{ay}\partial_{cy})f$$

$$=\left(\left\langle\sum_{\beta=1}^{4} h_\beta^2K_1\right\rangle+h_\beta^2\partial_{a\alpha}\eta^{b\alpha}\eta^{a\beta}\partial_{b\beta}+\langle A\rangle\right)f$$

$$=(h_\alpha^3\partial_{a\alpha}\eta^{a\alpha}+h_y^2\partial_{x\alpha}\eta^{y\alpha}\eta^{x\beta}\partial_{y\beta})f$$

$$=\left[\left\langle\sum_{\alpha=1}^{4} h_\alpha^3(4+h_\alpha)\right\rangle+\left\langle\sum_{x<y=1}^{4} h_y^2(h_y-h_x)\right\rangle\right]f$$

$$Af=\left(h_y\partial_{a\alpha}\eta^{b\alpha}\eta^{cx}\eta^{ay}\partial_{cy}\partial_{bx}+\sum_{x<y} h_y\partial_{a\alpha}\eta^{b\alpha}\eta^{ay}\partial_{by}\right)f$$

$$=\left(h_yh_x\partial_{ax}\eta^{bx}\eta^{ay}\partial_{by}+h_y\partial_{aw}\eta^{bw}\eta^{cx}\eta^{ay}\partial_{bx}\partial_{cy}\right.$$

$$\left.-\sum_{x<y} h_xh_y\partial_{ay}\eta^{ay}+\sum_{x<y} h_y^2\partial_{ay}\eta^{ay}+\sum_{x<y} h_y\partial_{az}\eta^{bz}\eta^{ay}\partial_{by}\right)f$$

$$(w<x<y, z<y)$$

$$=\left[\left\langle\sum_{x<y=1}^{4} h_y^2(h_y+4)\right\rangle+\left\langle\sum_{z,x<y=1} h_y(h_y-h_z)\right\rangle\right.$$

$$-\left\langle\sum_{x<y=1}^{4} h_xh_y(h_y+4)\right\rangle+\left\langle\sum_{x<y=1}^{4} h_xh_y(h_y-h_x)\right\rangle$$

$$\left.+\left\langle\sum_{w<x<y=1} h_y(h_w-h_x)\right\rangle\right]f$$

$$Bf = \left(\sum_{y<z} K_a^b \partial_{b\beta} \eta^{c\beta} \eta^{az} \partial_{cz} + K_a^b \partial_{b\beta} \eta^{c\beta} \eta^{dy} \eta^{az} \partial_{cy} \partial_{dz} \right) f$$

$$= \left(\sum_{y<z} h_y K_a^b \partial_{bz} \eta^{az} + \sum_{y<z} K_a^b \partial_{bx} \eta^{cx} \eta^{az} \partial_{cz} + \langle D \rangle \right) f$$

$$= \left(\left\langle \sum_{y<z=1} h_y K_1 \right\rangle + \sum_{y<z} h_z \partial_{a\alpha} \eta^{b\alpha} \eta^{az} \partial_{bz} + \langle C \rangle \right) f$$

$$Bf = \left\langle \sum_{y<z}^{4} h_z^2 \partial_{az} \eta^{az} + \sum_{y<z} h_z \partial_{ax} \eta^{bx} \eta^{az} \partial_{bz} \right\rangle f$$

$$= \left[\left\langle \sum_{y<z=1}^{4} h_z^2 (h_2 + 4) \right\rangle + \left\langle \sum_{x,y<z=1}^{4} h_z (h_z - h_x) \right\rangle \right] f$$

$$Cf = \left(\sum_{x,y<z}^{4} \partial_{a\alpha} \eta^{b\alpha} \eta^{az} \partial_{bz} + \sum_{y<z}^{4} \partial_{a\alpha} \eta^{b\alpha} \eta^{cx} \eta^{dz} \partial_{cz} \partial_{bx} \right) f$$

$$= \left(\sum_{x,y<z}^{y<z} h_z \partial_{az} \eta^{az} + \sum_{x,y<z}^{4} \partial_{aw} \eta^{bw} \eta^{az} \partial_{bz} + \sum_{y<z} \partial_{a\alpha} \eta^{cx} \eta^{az} \right.$$

$$\left. \partial_{cz} \eta^{b\alpha} \partial_{bx} - \sum_{y<z} \partial_{az} \eta^{bx} \eta^{az} \partial_{bx} \right) f \qquad (w<z)$$

$$= \left[\left\langle \sum_{x,y<z=1}^{4} h_z (h_z + 4) \right\rangle + \left\langle \sum_{w,x,y<z=1}^{4} (h_z - h_w) \right\rangle \right.$$

$$+ \sum_{y<z}^{4} h_x \partial_{ax} \eta^{cx} \eta^{az} \partial_{cz} + \sum_{y<z}^{4} \partial_{aw} \eta^{cx} \eta^{az} \partial_{cz} \eta^{bw} \partial_{bx}$$

$$- \left\langle \sum_{x,y<z=1}^{4} h_x (h_z + 4) \right\rangle f \qquad (w<x)$$

$$= \left[\left\langle \sum_{x,y<z=1}^{4} h_x (h_z - h_x) \right\rangle + \sum_{w<x,y<z} \eta^{cx} \partial_{bx} \eta^{bz} \partial_{cz} \right.$$

$$\left. + \sum_{y<z} \eta^{cx} \partial_{bx} \eta^{az} \partial_{cz} \eta^{bw} \partial_{aw} \right] f$$

$$= \left[\left\langle \sum_{w<x;\, x,y<z=1}^{4} (h_w - h_x) \right\rangle \right] f$$

$$Df = \left(h_y K_a^b \partial_{by} \eta^{dy} \eta^{az} \partial_{dz} - \sum_{y<z} h_y K_a^b \partial_{bz} \eta^{az} \right.$$

$$\left. + K_a^b \partial_{bx} \eta^{cx} \eta^{dy} \eta^{az} \partial_{cy} \partial_{dz} \right) f \qquad (x<y)$$

$$= \left(\sum_{y<z} h_y K_a^b \eta^{az} \partial_{bz} + h_y \partial_{a\alpha} \eta^{b\alpha} \eta^{dy} \eta^{az} \partial_{dz} \partial_{by} - \sum_{y<z} h_y K_a^b \eta^{az} \partial_{bz} \right.$$

$$- \left\langle \sum_{y<z=1} h_y K_1 \right\rangle + \sum_{x<y} \partial_{a\alpha} \partial^{b\alpha} \eta^{dy} \eta^{az} \partial_{by} \partial_{dz}$$

$$\left. + \partial_{a\alpha} \eta^{b\alpha} \eta^{cx} \eta^{dy} \eta^{az} \partial_{cy} \partial_{dz} \partial_{bx} \right) f$$

$$Df = \left(- \sum_{y<z} h_y^2 \partial_{az} \eta^{az} + h_y^2 \partial_{ay} \eta^{by} \eta^{az} \partial_{bz} + h_y \partial_{ax} \eta^{bx} \eta^{cy} \eta^{az} \partial_{by} \partial_{cz} \right.$$

$$+ \sum_{x<y} \partial_{aw} \eta^{bw} \eta^{cy} \eta^{az} \partial_{by} \partial_{cz} - \sum_{x<y} h_y \partial_{az} \eta^{az} + \sum_{x<y} h_y \partial_{ay}$$

$$(w<y, x<y)$$

$$\eta^{by} \eta^{az} \partial_{bz} + \sum_{x<y<z} h_y \partial_{az} \eta^{az} - \sum_{x<z} h_x \partial_{ay}$$

$$\eta^{by} \eta^{az} \partial_{bz} + h_x \partial_{ax} \eta^{cx} \eta^{dy} \eta^{az} \partial_{cy} \partial_{dz} +$$

$$\left. \partial_{av} \eta^{bv} \eta^{cx} \eta^{dy} \eta^{az} \partial_{bx} \partial_{cy} \partial_{dz} \right) f \qquad (v<x)$$

$$= \left[- \left\langle \sum_{y<z=1}^{4} h_z^2 (h_z + 4) \right\rangle + \left\langle \sum_{y<z=1}^{4} h_y^2 (h_z - h_y) \right\rangle \right.$$

$$+ \left\langle \sum_{x<y<z=1}^{4} h_y (h_z + 4) \right\rangle + \left\langle \sum_{w,x<y;\, y<z=1}^{4} (h_w - h_y) \right\rangle$$

$$- \left\langle \sum_{x<y<z=1}^{4} h_y (h_z + 4) \right\rangle + \left\langle \sum_{x<y<z=1}^{4} h_y (h_z - h_x) \right\rangle$$

$$+ \left\langle \sum_{v<x<y<z=1}^{4} (h_y - h_w) \right\rangle f$$

$$K_4 f = [1024 + 85h_1 + 155h_2 + 258h_3 + 499h_4 + h_1(8h_2 + 10h_3 + 12h_4) + h_2(12h_3 + 14h_4) + 16h_3h_4 + 27h_1^2 + 53h_2^2 + 91h_3^2 + 144h_4^2 + h_1(h_2^2 + h_3^2 + h_4^2) + h_2(h_3^2 + h_4^2) + h_3h_4^2 + h_1^2(h_2 + h_3 + h_4) + h_2^2(h_3 + h_4) + h_3^2h_4 + 7h_1^3 + 11h_2^3 + 15h_3^3 + 19h_4^3 + h_1^4 + h_2^4 + h_3^4 + h_4^4] f \qquad \text{(A7)}$$

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